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⑥ A Stability Investigation for an Incompressible
Simple Fluid with Fading Memory

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by

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ABSTRACT

The nonlinear equations of motion for an incompressible simple fluid, occupying a fixed bounded container, are formulated on the basis of the "finite-linear" viscoelastic theory for materials with fading memory; this formal boundary-initial value problem is then viewed as a nonlinear abstract evolution equation on a certain Hilbert space. It is shown that a linearized version of this evolution equation is associated with a linear dynamical system on this Hilbert space, and several stability and asymptotic behavior results for this linearized problem are proved through the use of Liapunov stability methods. On the assumption that the original nonlinear evolution equation also is associated with some dynamical system on the same space, it is shown that the rest condition of the fluid is stable and all motions are bounded. The Liapunov function employed for this purpose can be interpreted as a mechanical energy function for the fluid.

Key words:

   simple fluid, viscoelastic, fading memory, stability, Liapunov function, dynamical system.

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1. Introduction

In this paper we study a boundary-initial value problem describing the
motions of an incompressible simple fluid with fading memory, assuming a
"finite-linear" constitutive equation as formulated by Coleman and Noll (1961),
(1964). Stability results for linearized versions of this problem have been
obtained by Craik (1968) and Joseph (1974), who employed spectral analysis
for this purpose; more recently, Slemrod (1976), (1978) has performed a
stability analysis for one such linearized problem by using the ideas of
dynamical systems theory. We refer the reader to Slemrod (1976) for a critique
of the spectral analysis approach.

Here we are primarily interested in the highly nonlinear equations of
motion which result from a careful formulation of the general problem described
above, employing only physically reasonable assumptions. Our most important
assumption is that these nonlinear equations do lead to a dynamical system
on an appropriate state space, and we motivate this assumption by also studying
a linearized version of the problem. Although our linearized problem is
closely related to the linear problem considered by Slemrod (1976), (1978), we
are able to show that certain of the stability results for our linearized problem
do carry over to the original nonlinear problem, whereas those of Slemrod (1976),
(1978), Craik (1968), and Joseph (1974) apparently do not. Our approach
here is based entirely on the ideas of dynamical systems theory.

In §2 we formulate the basic equations of motion for a simple incompressible
fluid, based on the "finite-linear" viscoelastic constitutive equation of
Coleman and Noll (1961). Assuming that for all time $t \geq 0$ (but not $t < 0$) the
fluid is incompressible and fills a fixed bounded container, we obtain a

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formal boundary-initial value problem with history dependence of possibly
infinite duration. Choosing a state space equipped with a "fading memory"
norm (see Coleman and Mizel (1966)), we view the formal problem as an abstract
evolution equation on this space; tractability of this problem seems to be highly
dependent on selection of the appropriate state space.

As we are unable to prove that our nonlinear evolution equation leads
to a dynamical system, we consider a linearized version in §3. There we show
that this linear evolution equation does lead to a dynamical system (on the
same space), and we obtain certain results on stability, asymptotic stability,
and exponential decay of motions. Although our linear problem is closely
related to those of Craik (1968), Joseph (1974), and Slemrod (1976), (1978),
we make fewer assumptions regarding the deformation history. Finally, in §4
we assume that, in a certain sense, the original nonlinear problem is related
to a dynamical system; we then are able to prove that the rest condition is
stable and every motion is bounded in terms of the initial state of the fluid.
Our results are obtained through the use of a Liapunov function (see Hale (1969))
which we interpret as a mechanical energy function for the fluid.

2. Formulation of the Problem

We consider an incompressible simple fluid occupying a bounded domain $\Omega \in R^3$, with C^1-smooth boundary Γ, for all time $t \in R^+ \equiv [0,\infty)$. Following the development of Noll (1958) and Coleman and Noll (1961), (1964), we formulate in this section a set of nonlinear equations of motion based on the "finite-linear" viscoelastic theory.

For this purpose, consider an arbitrary fluid particle that has position $\eta = (\eta_1, \eta_2, \eta_3) \in \Omega$ at time $t \in R^+$, letting $X(\tau; \eta, t) = (X_1, X_2, X_3) \in R^3$ and $\rho(\tau; \eta, t) \in R^+$ denote its position and mass density, respectively, at time $\tau \in R$ (note that $t \geq 0$ but τ may be negative). The _relative deformation gradient_ $F(\tau; \eta, t)$ is the second order tensor whose components are given by $f_{ij}(\tau; \eta, t) \equiv \partial_j X_i(\tau; \eta, t)$, where $\partial_j \equiv \dfrac{\partial}{\partial \eta_j}$. It is known that

$$(2.1) \qquad \rho(t; \eta, t) = \rho(\tau; \eta, t) \det F(\tau; \eta, t)$$

for all $(\tau, \eta, t) \in R \times \Omega \times R^+$.

Following Coleman and Noll (1961), we denote by $C(\tau; \eta, t)$ the _relative right Cauchy-Green tensor_ with components $c_{ij} \equiv f_{ki} f_{kj}$. Here and in the sequel we employ the convention of summation on repeated indices. Under the assumption of isotropy and homogeneity (see Coleman and Noll (1961), (1964)), the basic constitutive equations of the "finite-linear" theory of viscoelasticity state that, for a compressible simple fluid, the components of the stress tensor $S(\eta, t)$ are given by

$$s_{ij}(\eta, t) = - \left\{ \widetilde{p}(\rho(t; \eta, t)) + \int_0^\infty \widetilde{h}(s, \rho(t; \eta, t))[c_{kk}(t - s; \eta, t) - \delta_{kk}]ds \right\} \delta_{ij}$$

$$- \int_0^\infty \widetilde{m}(s, \rho(t; \eta, t))[c_{ij}(t - s; \eta, t) - \delta_{ij}]ds \quad , \quad (\eta, t) \in \Omega \times R^+ \quad ,$$

where δ_{ij} denotes the Kronecker delta and $\tilde{p}$, $\tilde{h}$, $\tilde{m}$ are scalar-valued material functions (see equation 5.18 in Coleman and Noll (1961)). Here we assume the fluid to be incompressible for $\tau \in R^+$; hence, the density is constant and we set $\rho(\tau; \eta, t) \equiv 1$ for $\tau \in R^+$, $\eta \in \Omega$, $t \in R^+$, without loss of generality. Under this assumption the above expression for stress must be replaced by

$$(2.2) \quad s_{ij}(\eta, t) = - p(\eta, t)\delta_{ij} - \int_0^\infty m(s)[c_{ij}(t - s; \eta, t) - \delta_{ij}]ds \quad , \quad (\eta, t) \in \Omega \times R^+ \quad ,$$

where $p(\eta, t)$ is a constitutively indeterminate quantity and $m : R^+ \to R$ is a material function. We assume that this function satisfies the following conditions:

$$\text{a)} \quad m \in C^1(R^+) \cap \mathcal{L}_1(R^+) \quad ,$$

(2.3) b) m is nonnegative and nonincreasing on R^+ ,

$$\text{c)} \quad m(s) > 0 \text{ for } s \in [0, r); \ m(s) = 0 \text{ for } s \notin [0, r) \quad ,$$

where $r > 0$ may be infinite. This mild assumption follows from the concept of "fading memory" as stated by Coleman and Noll (1961).

We remark that the validity of equation (2.2) depends on the assumption of incompressibility for $\tau \in R^+$, but does not presume incompressibility for $\tau < 0$. Furthermore, we note that, by equation (2.1), the assumption $\rho(\tau; \eta, t) = 1$ for $\tau \in R^+$ implies that

$$\det F(\tau; \eta, t) = 1/\rho(\tau; \eta, t) \quad , \quad \tau \in R \quad ,$$

(2.4)

$$\det F(\tau; \eta, t) = 1 \quad , \quad \tau \in R^+ \quad ,$$

for all $(\eta, t) \in \Omega \times R^+$.

We assume the fluid container to be fixed in some inertial system, and that all particle positions are measured relative to this inertial system. We also assume the body force (per unit mass) at (η, t) to be the gradient $\partial_i q(\eta, t)$ of a known external potential $q : \Omega \times R^+ \to R$. Consequently, the balance law for linear momentum takes the form

$$(2.5) \quad \left. \frac{\partial^2}{\partial \tau^2} X_i(\tau; \eta, t) \right|_{\tau = t} = \partial_i q(\eta, t) + \partial_j s_{ij}(\eta, t) \quad , \quad (\eta, t) \in \Omega \times R^+ \quad .$$

It is convenient to introduce some new notation; let

$$v(\eta, t) \equiv \left. \frac{\partial}{\partial \tau} X(\tau; \eta, t) \right|_{\tau = t} \quad ,$$

$$(2.6)$$

$$u(s, \eta, t) \equiv X(t - s; \eta, t) - X(t; \eta, t) = X(t - s; \eta, t) - \eta \quad ,$$

for all $(s, \eta, t) \in R^+ \times \Omega \times R^+$. For a particle with position $\eta \in \Omega$ at time $t \in R^+$, $v(\eta, t)$ represents its velocity at time t and $u(s, \eta, t) + \eta$ its position at time $t - s \leq t$. Some straightforward manipulations show that

$$(2.7) \qquad \left. \frac{\partial^2}{\partial \tau^2} X_i(\tau; \eta, t) \right|_{\tau = t} = v_j(\eta, t) \partial_j v_i(\eta, t) + \frac{\partial}{\partial t} v_i(\eta, t) \quad ,$$

$$(2.8) \qquad \frac{\partial}{\partial t} u_i(s, \eta, t) = - v_j(\eta, t)[\delta_{ij} + \partial_j u_i(s, \eta, t)] - \frac{\partial}{\partial s} u_i(s, \eta, t) \quad ,$$

$$(2.9) \qquad u_i(0, \eta, t) = 0 \quad , \quad \left. \frac{\partial}{\partial s} u_i(s, \eta, t) \right|_{s = 0} = - v_i(\eta, t) \quad ,$$

$$(2.10) \quad c_{ij}(t - s; \eta, t) - \delta_{ij} = f_{ki}(t - s; \eta, t) f_{kj}(t - s; \eta, t) - \delta_{ij}$$

$$= \partial_i u_k(s, \eta, t) \cdot \partial_j u_k(s, \eta, t) + \partial_i u_j(s, \eta, t) + \partial_j u_i(s, \eta, t) \quad ,$$

for all $(s, \eta, t) \in R^+ \times \Omega \times R^+$. Since the first of (2.9) implies $\partial_j u_i(0, \eta, t) = 0$, relation (2.4) leads to the condition

$$\det\left|\partial_j u_i(s,\eta,t) + \delta_{ij}\right| = \det\left|\partial_j u_i(0,\eta,t) + \delta_{ij}\right|$$

$$= \det\left|\delta_{ij}\right| = 1 \quad ,$$

for all $t \geq s \geq 0$, $\eta \in \Omega$; hence, we also have

$$(2.11) \qquad 0 = \left.\left(\frac{\partial}{\partial s} \det\left|\partial_j u_i(s,\eta,t) + \delta_{ij}\right|\right)\right|_{s=0} = \left.\frac{\partial}{\partial s} \partial_i u_i(s,\eta,t)\right|_{s=0}$$

$$= -\partial_i v_i(\eta,t) \quad , \quad (\eta,t) \in \Omega \times \mathbb{R}^+ \quad ,$$

where we have used the second of (2.9).

Upon collecting (2.2), (2.3), (2.5)-(2.11), and making a final physical assumption that the velocity of the fluid is zero at the boundary Γ, the fluid motion is seen to be described by the formal evolution equations

$$(2.12) \quad \frac{\partial}{\partial t} v_i(\eta,t) = -v_j(\eta,t)\partial_j v_i(\eta,t) - \partial_i p(\eta,t) + \partial_i q(\eta,t)$$

$$-\partial_j \int_0^r m(s)\left[\partial_i u_k(s,\eta,t)\cdot\partial_j u_k(s,\eta,t) + \partial_i u_j(s,\eta,t)\right.$$

$$\left. + \partial_j u_i(s,\eta,t)\right]ds \quad , \quad \eta \in \Omega \quad ,$$

$$(2.13) \qquad \frac{\partial}{\partial t} u_i(s,\eta,t) = -v_j(\eta,t)\left[\delta_{ij} + \partial_j u_i(s,\eta,t)\right]$$

$$-\frac{\partial}{\partial s} u_i(s,\eta,t) \quad , \quad (s,\eta) \in \mathbb{R}^+ \times \Omega \quad ,$$

subject to

$$\partial_i v_i(\eta,t) = 0 \quad , \quad \eta \in \Omega \quad ,$$

$$(2.14) \qquad u_i(0,\eta,t) = 0 = v_i(\eta,t) + \left.\frac{\partial}{\partial s} u_i(s,\eta,t)\right|_{s=0} \quad , \quad \eta \in \Omega \quad ,$$

$$v_i(\eta,t) = 0 \quad , \quad \eta \in \Gamma \quad ,$$

for all $t \in \mathbb{R}^+$, with prescribed initial data $(v_i(\eta,0), u_i(s,\eta,0))$ satisfying (2.14).

The unknown pressure gradient in (2.12) creates certain difficulties, and we now employ a well known device for removing this term (see Fujita and Kato (1964)). Consider the Hilbert space $(\mathcal{L}_2(\Omega))^3$, equipped with the usual inner product, and let $\Pi : (\mathcal{L}_2(\Omega))^3 \to (\mathcal{L}_2(\Omega))^3$ be the orthogonal projection whose range $\mathfrak{R}(\Pi)$ is the closure in $(\mathcal{L}_2(\Omega))^3$ of $\{\tilde{v} \in (\mathbb{C}_o^\infty(\Omega))^3 | \partial_i \tilde{v}_i(\eta) = 0, \; \eta \in \Omega\}$. The range of Π is orthogonal to the closure of the set of elements of $(\mathcal{L}_2(\Omega))^3$ which are gradients of scalar-valued functions. Writing Π as a symmetric second order tensor π_{ij} and applying it to (2.12), we find that

$$(2.15) \quad \frac{\partial}{\partial t} v_i(\eta, t) = - \pi_{i\ell} v_j(\eta, t) \partial_j v_\ell(\eta, t)$$

$$- \pi_{i\ell} \partial_j \int_0^r m(s) [\partial_\ell u_k(s, \eta, t) \cdot \partial_j u_k(s, \eta, t) + \partial_j u_\ell(s, \eta, t)] ds \; , \quad \eta \in \Omega \; .$$

Once a solution (v, u) of equations (2.13)-(2.15) is found, the corresponding pressure gradient $\partial_i p(\eta, t)$ can be recovered through equation (2.12), or through the equation obtained by applying the projection $I - \Pi$ to (2.12).

We wish to emphasize two points about our physical assumptions. It should be noted that conditions (2.14) do not require the fluid to have always occupied the domain Ω in the past; i.e., for $s > t \geq 0$, we have not assumed that

$$(2.16) \quad \begin{aligned} u(s, \eta, t) + \eta \in \Omega \; , \quad \eta \in \Omega \; , \\ u(s, \eta, t) = 0 \quad , \quad \eta \in \Gamma \; . \end{aligned}$$

Secondly, conditions (2.14) do not require the fluid to have always been incompressible in the past; i.e., for $s > t \geq 0$, we have not assumed that

(2.17)
$$\det\left|\partial_j u_i(s,\eta,t) + \delta_{ij}\right| = 1 \quad , \quad \eta \in \Omega \quad .$$

It is apparent that at least the first of these non-assumptions is physically important.

We now wish to put (2.13)-(2.15) in the form of an abstract evolution equation on an appropriate state space. For this purpose, first consider the linear space S of pairs (v,u) in $(C_o^\infty(\Omega))^3 \times \bigcup_{0 < \beta < r} (P_\beta)^3$, where

$$P_\beta \equiv \left\{ w \in C^\infty([0,r) \times \overline{\Omega}) \,\middle|\, w(s,\eta) = 0 \text{ for } (s,\eta) \in [\beta,r) \times \overline{\Omega} \right\} \quad ,$$

such that $v \in R(\Pi)$ and $u(0,\eta) = 0 = v(\eta) + \dfrac{\partial}{\partial s} u(s,\eta)\Big|_{s=0}$ for all $\eta \in \overline{\Omega}$. Recalling conditions (2.3) on $m(s)$, where $r > 0$ might be infinite, we define

$$\|(v,u)\|_{\mathcal{X}}^2 \equiv \int_\Omega \left\{ v_i(\eta)v_i(\eta) + \int_0^r m(s)\partial_k u_i(s,\eta)\cdot\partial_k u_i(s,\eta)\,ds \right\} d\Omega$$

and we let $\mathcal{X}$ denote the Hilbert space obtained by $\|\cdot\|_{\mathcal{X}}$-completion of S.

Also, for $x = (v,u) \in S$ we define

$$Nx \equiv (w,z) \quad ,$$

(2.18)
$$w_i(\eta) \equiv -\pi_{i\ell} v_j(\eta)\partial_j v_\ell(\eta)$$

$$-\pi_{i\ell}\partial_j \int_0^r m(s)[\partial_\ell u_k(s,\eta)\cdot\partial_j u_k(s,\eta) + \partial_j u_\ell(s,\eta)]ds \quad ,$$

$$z_i(s,\eta) \equiv -v_j(\eta)\left[\delta_{ij} + \partial_j u_i(s,\eta)\right] - \frac{\partial}{\partial s} u_i(s,\eta) \quad ,$$

and we consider a metric on S defined by $d_N(x,\hat{x}) \equiv \|x - \hat{x}\|_{\mathcal{X}} + \|Nx - N\hat{x}\|_{\mathcal{X}}$, $x,\hat{x} \in S$. We denote by $\mathcal{D}(N)$ the d_N-completion of S, and we let $N : (\mathcal{D}(N) \subset \mathcal{X}) \to \mathcal{X}$ be the operator defined on $\mathcal{D}(N)$ by (2.18). Clearly, $\mathcal{D}(N)$ is dense in $\mathcal{X}$, and the completeness of $(\mathcal{D}(N),d_N)$ implies that N is closed as a (nonlinear) mapping from $\mathcal{X}$ to $\mathcal{X}$.

Within this setting we can replace (2.13)-(2.15) by an abstract evolution equation on $\mathfrak{X}$, given by

$$\dot{x}(t) = Nx(t) \quad \text{a.e.} \quad t \in \mathbb{R}^+ \quad ,$$

(2.19)

$$x(0) = x_o \in \mathfrak{X} \quad .$$

This highly nonlinear evolution equation is very difficult to analyze, even as to existence and uniqueness of solutions. However, for $x_o = 0$, we notice that equation (2.19) does admit as a solution the rest condition, $x(t) = 0$ for all $t \in \mathbb{R}^+$. In the following section we study a linearized version of this equation and show that it leads to a stable linear dynamical system on $\mathfrak{X}$.

3. A Linearized Problem

We are unable to prove that the nonlinear abstract evolution equation
(2.19) is related to any dynamical system (see Hale (1969)); in §4 we will simply
assume that such a relationship does exist. To suggest the plausibility of
this forthcoming assumption, we now perform an appropriate linearization
of (2.19) (on the same Hilbert space $\mathcal{X}$) and show that the resulting abstract
evolution equation does lead to a (linear) dynamical system on $\mathcal{X}$. We remark
that this Hilbert space is different from the one used by Slemrod (1976), (1978)
for a similar but not identical linear problem.

Our first step is to return to the formal equations (2.13)-(2.15) and
delete all nonlinear terms; hence, conditions (2.14) are retained, while
(2.15) and (2.13) are replaced by

$$(3.1) \qquad \frac{\partial}{\partial t} v_i(\eta, t) = - \pi_{i\ell} \partial_j \int_0^r m(s) \partial_j u_\ell(s, \eta, t) ds \quad , \quad \eta \in \Omega \quad ,$$

$$(3.2) \qquad \frac{\partial}{\partial t} u_i(s, \eta, t) = - v_i(\eta, t) - \frac{\partial}{\partial s} u_i(s, \eta, t) \quad , \quad (s, \eta) \in R^+ \times \Omega \quad ,$$

for all $t \in R^+$. We continue to assume that $m(s)$ satisfies conditions (2.3).

Recalling the linear space S and the Hilbert space $\mathcal{X}$ defined in Section II,
we define, for $x = (v, u) \in S$,

$$Ax \equiv (w, z) \quad ,$$

$$(3.3) \qquad w_i(\eta) \equiv - \pi_{i\ell} \partial_j \int_0^r m(s) \partial_j u_\ell(s, \eta) ds \quad ,$$

$$z_i(s, \eta) \equiv - v_i(\eta) - \frac{\partial}{\partial s} u_i(s, \eta) \quad ,$$

and we consider a norm on S defined by $\|x\|_A \equiv \|x\|_\chi + \|Ax\|_\chi$, $x \in S$. We denote by $\mathcal{D}(A)$ the $\|\cdot\|_A$ - completion of S, and we let $A : (\mathcal{D}(A) \subseteq \chi) \to \chi$ be the linear operator defined on $\mathcal{D}(A)$ by (3.3). We see that $\mathcal{D}(A)$ is dense in χ and A is closed as a linear mapping from χ to χ.

The formal linear problem (3.1), (3.2), (2.14) leads us to consider a linear abstract evolution equation defined in the Hilbert space χ by

$$\dot{x}(t) = Ax(t) \quad , \quad t \in R^+ \quad ,$$

(3.4)

$$x(0) = x_o \in \mathcal{D}(A) \subseteq \chi \quad .$$

We wish to show that (3.4) is related to a linear dynamical system on χ, and that the motions of this dynamical system are (unique) solutions of (3.4) for $x_o \in \mathcal{D}(A)$.

We recall (see Hale (1969), Yosida (1978)) that a <u>dynamical</u> <u>system</u> $\{T(t)\}_{t \geq 0}$, on a metric space χ, is a family of continuous operators $T(t) : \chi \to \chi$ such that $T(\cdot)x : R^+ \to \chi$ is continuous, $T(0)x = x$, and $T(t+h)x = T(t)T(h)x$, for all $t,h \in R^+$, $x \in \chi$. The mapping $T(\cdot)x : R^+ \to \chi$ and the set $\gamma(x) \equiv \underset{t \geq 0}{\cup} T(t)x$ are called, respectively, the <u>motion</u> and the <u>positive</u> <u>orbit</u> corresponding to the initial <u>state</u> $x \in \chi$. A subset $\mathcal{G} \subseteq \chi$ is said to be <u>positive</u> <u>invariant</u> under $\{T(t)\}_{t \geq 0}$ if $\gamma(x) \subseteq \mathcal{G}$ for every $x \in \mathcal{G}$. A motion $T(\cdot)x$ is said to be <u>stable</u> if, given any $\epsilon > 0$, there exists a neighborhood $n_\delta(x)$ of radius $\delta > 0$ such that $y \in n_\delta(x)$ implies $T(t)y \in n_\epsilon(T(t)x)$ for all $t \in R^+$; $T(\cdot)x$ is <u>asymptotically</u> <u>stable</u> if it is stable and $T(t)y \to T(t)x$ as $t \to \infty$ for all y in some neighborhood of x. Furthermore, we recall that $\{T(t)\}_{t \geq 0}$ is called a <u>linear</u> <u>dynamical</u> <u>system</u> if χ is a Banach space and $T(t) : \chi \to \chi$ is a linear operator for every $t \in R^+$; the <u>infinitesimal</u> <u>generator</u> $B : (\mathcal{D}(B) \subseteq \chi) \to \chi$ of such a linear dynamical system is defined by

$$Bx \equiv \lim_{t \searrow 0} \frac{1}{t} [T(t)x - x] \quad ,$$

where $\mathcal{D}(B)$ is the set of $x \in \mathcal{X}$ such that this limit exists. If B is the infinitesimal generator of the linear dynamical system $\{T(t)\}_{t \geq 0}$ defined on $\mathcal{X}$, then for every $x_o \in \mathcal{D}(B)$ the motion $T(\cdot)x_o$ is the unique strong solution of the equation $\dot{x}(t) = Bx(t)$, $t \geq 0$, for the initial state $x(0) = x_o$.

With this terminology, we can prove the following result for our linear abstract evolution equation (3.4).

Theorem 3.1: For m(s) satisfying (2.3), the linear operator $A : (\mathcal{D}(A) \subseteq \mathcal{X}) \rightarrow \mathcal{X}$ is the infinitesimal generator of a linear dynamical system $\{T(t)\}_{t \geq 0}$ on $\mathcal{X}$, with $\|T(t)x\|_{\mathcal{X}} \leq \|x\|_{\mathcal{X}}$ for all $t \in \mathcal{R}^+$, $x \in \mathcal{X}$.

Proof: As $\mathcal{D}(A)$ is dense in $\mathcal{X}$, for the theorem to be proven it is sufficient (see Yosida (1978)) to show that $-A$ is $\|\cdot\|_{\mathcal{X}}$ - accretive and that the range $\mathcal{R}(I - A) = \mathcal{X}$. Using the natural inner product for $\mathcal{X}$, and considering arbitrary $x = (u,v) \in \mathcal{D}(A)$, we see that

$$\langle x, Ax \rangle = - \int_{\Omega} v_i(\eta) \cdot \pi_{i\ell} \partial_j \int_0^r m(s) \partial_j u_\ell(s,\eta) \, ds \, d\Omega$$

$$- \int_{\Omega} \int_0^r m(s) \partial_k u_i(s,\eta) \cdot \partial_k [v_i(\eta) + \frac{\partial}{\partial s} u_i(s,\eta)] \, ds \, d\Omega$$

$$= - \int_{\Omega} v_\ell(\eta) \cdot \partial_j \int_0^r m(s) \partial_j u_\ell(s,\eta) \, ds \, d\Omega$$

$$- \int_{\Omega} \partial_k v_i(\eta) \cdot \int_0^r m(s) \partial_k u_i(s,\eta) \, ds \, d\Omega$$

$$- \frac{1}{2} \int_{\Omega} \int_0^r m(s) \frac{\partial}{\partial s} [\partial_k u_i(s,\eta) \cdot \partial_k u_i(s,\eta)] \, ds \, d\Omega$$

$$= \frac{1}{2} \int_{\Omega} \int_0^r m'(s) \cdot \partial_k u_i(s,\eta) \cdot \partial_k u_i(s,\eta) \, ds \, d\Omega \leq 0 \quad ;$$

hence, $-A$ is accretive.

As $-A$ is accretive, it follows that $I - A$ possesses a bounded inverse defined on $R(I - A)$. As A is a closed operator, it follows that $(I - A)^{-1}$ is closed and bounded; hence, $R(I - A) = \mathcal{X}$ if $R(I - A)$ is dense in $\mathcal{X}$. With this in mind, let (w,z) be a fixed but arbitrary element of the dense set $S \subset \mathcal{X}$, and consider the equations

$$v_i(\eta) + \pi_{i\ell}\partial_j \int_0^r m(s) \cdot \partial_j u_\ell(s,\eta)ds = w_i(\eta) \quad ,$$

(3.5)

$$u_i(s,\eta) + v_i(\eta) + \frac{\partial}{\partial s} u_i(s,\eta) = z_i(s,\eta) \quad .$$

If these equations can be shown to have a solution (v,u) belonging to $\mathcal{D}(A)$ we will have shown that $R(I - A) \supset S$, and by the denseness of S in $\mathcal{X}$ it will follow that $R(I - A) = \mathcal{X}$. Formally, the second of equations (3.5) implies that

$$(3.6) \qquad u_i(s,\eta) = -(1 - e^{-s})v_i(\eta) + e^{-s}\int_0^s e^{\xi}z_i(\xi,\eta)d\xi \quad ;$$

hence, $u_i(0,\eta) = 0$ and $\left.\frac{\partial}{\partial s} u_i(s,\eta)\right|_{s=0} = -v_i(\eta)$ for all $\eta \in \Omega$. Defining

$$\alpha \equiv \int_0^r m(s)(1 - e^{-s})ds \quad ,$$

(3.7)

$$\hat{w}_i(\eta) \equiv w_i(\eta) - \pi_{i\ell}\partial_j \int_0^r m(s)e^{-s}\partial_j \int_0^s e^{\xi}z_\ell(\xi,\eta)d\xi\,ds \quad ,$$

we note that $\hat{w} \in R(\Pi) \cap (C^\infty(\overline{\Omega}))^3$, and (2.3) implies that $0 < \alpha < \infty$. Using (3.6) and (3.7), the first of equations (3.5) becomes

$$(3.8) \qquad v_i(\eta) - \alpha\pi_{i\ell}\partial_j\partial_j v_\ell(\eta) = \hat{w}_i(\eta) \quad .$$

It follows from elliptic theory (see Mizohata (1973)) that (3.8) admits a solution $v \in \mathcal{R}(\Pi) \cap (\mathbb{C}^\infty(\bar{\Omega}))^3$ such that $v_i(\eta) = 0$ for $\eta \in \Gamma$; inserting this v in (3.6) we see that $u \in (\mathbb{C}^\infty([0,r) \times \bar{\Omega}))^3$. If $r < \infty$, we now can conclude that $(v,u) \in \mathcal{D}(A)$. If $r = \infty$, let $0 < \beta < \infty$ and note that, by (3.5) and through the same argument used in proving that $-A$ was accretive,

$$\|(w,z)\|_\chi^2 \geq \int_\Omega \left\{ w_i(\eta) w_i(\eta) + \int_0^\beta m(s) \partial_j z_i(s,\eta) \cdot \partial_j z_i(s,\eta) ds \right\} d\Omega$$

$$\geq \int \left\{ v_i(\eta) v_i(\eta) + \int_0^\beta m(s) \partial_j u_i(s,\eta) \cdot \partial_j u_i(s,\eta) ds \right\} d\Omega$$

$$+ \int_\Omega \left\{ \pi_{i\ell} \partial_j \int_0^\beta m(s) \partial_j u_\ell(s,\eta) ds \cdot \pi_{ik} \partial_h \int_0^\beta m(s) \partial_h u_k(s,\eta) ds \right.$$

$$+ \int_0^\beta m(s) \partial_j \left[v_i(\eta) + \frac{\partial}{\partial s} u_i(s,\eta) \right] \cdot \partial_j \left[v_i(\eta) + \frac{\partial}{\partial s} u_i(s,\eta) \right] ds \right\} d\Omega .$$

Letting $\beta \to r = \infty$, we see that $(u,v) \in \mathcal{D}(A)$, with $\|(u,v)\|_\chi^2 + \|A(u,v)\|_\chi^2 \leq \|(w,z)\|_\chi^2$. Hence, for any $r \in (0,\infty]$, we have shown that (3.5) has a solution $(v,u) \in \mathcal{D}(A)$ for every $(w,z) \in S$, and thus we conclude that $\mathcal{R}(I - A) = \chi$. The proof is complete.

Theorem 3.1 shows that our abstract linear evolution equation (3.4) has a unique solution for every $x_o \in \mathcal{D}(A)$. Furthermore, as the dynamical system $\{T(t)\}_{t \geq 0}$ is linear, this theorem implies that all positive orbits are bounded and every motion is stable.

For $r < \infty$ the injection $(\mathcal{D}(A), \|\cdot\|_A) \hookrightarrow \chi$ is compact, and it follows that $(I - A)^{-1}$ is a compact operator. Hence, if $r < \infty$, we may now conclude that all positive orbits are precompact (see Dafermos and Slemrod (1973)), and this fact enables us to prove the following result.

Theorem 3.2: For $m(s)$ satisfying (2.3) with $r < \infty$, every motion of $\{T(t)\}_{t \geq 0}$ is asymptotically stable.

Proof: Defining $V : \mathcal{X} \to \mathcal{R}$ as $V(x) = \langle x, x \rangle$, and defining $\dot{V} : \mathcal{X} \to \bar{\mathcal{R}}$ by

$$\dot{V}(x) \equiv \lim_{t \searrow 0} \inf \frac{1}{t} \left[V(T(t)x) - V(x) \right] , \quad x \in \mathcal{X} ,$$

we see that $\dot{V}(x) = 2\langle x, Ax \rangle$ for $x \in \mathcal{D}(A)$; therefore, our accretivity argument shows that

$$\dot{V}(x) = \int_{\Omega} \int_0^r m'(s) \cdot \partial_k u_i(s, \eta) \cdot \partial_k u_i(s, \eta) \, ds \, d\Omega \cong - W(x) \leq 0$$

for all $x = (v, u) \in \mathcal{D}(A)$. By Theorem 3.9 of Walker (1976), it follows that $\dot{V}(x) \leq - W(x) \leq 0$ for all $x \in \mathcal{X}$. Hence, V is a Liapunov function on $\mathcal{X}$ (see Hale (1969)).

Conditions (2.3) on $m(s)$ imply that $m'(s) < 0$ for all s in some nonempty open set $\mathcal{I} \subset (0, r)$; consequently,

$$\{x \in \mathcal{X} \,|\, \dot{V}(x) = 0\} \subset \left\{ (v, u) \in \mathcal{X} \,\Big|\, \int_{\Omega} \partial_k u_i(s, \eta) \cdot \partial_k u_i(s, \eta) \, d\Omega = 0 \quad \text{a.e.} \quad s \in \mathcal{I} \right\}$$

Using equation (3.4), it is not difficult to see that the largest positive invariant subset $\mathcal{M}^+$ of $\{x \in \mathcal{X} \,|\, \dot{V}(x) = 0\}$ is $\mathcal{M}^+ = \{0\}$. As all positive orbits are precompact, LaSalle's Invariance Principle (see Hale (1969)) now implies that $T(t)x \to 0$ as $t \to \infty$, for every $x \in \mathcal{X}$; hence, $x = 0$ is an asymptotically stable equilibrium. By the linearity of the dynamical system, it follows that all motions are asymptotically stable, and the proof is complete.

In two recent papers Slemrod (1976), (1978) has used similar methods to study a problem very closely related to our formal linear problem (3.1), (3.2),

(2.14). Rather than using the pair (v,u) to represent the state of the system,
Slemrod chose the pair $\left(v, -\frac{\partial}{\partial s} u\right)$. He also placed additional restrictions
on the history   corresponding to (2.16) and a linearization of (2.17), namely

$$u(s,\eta) + \eta \in \Omega \quad , \quad (s,\eta) \in R^+ \times \Omega \quad ,$$

(3.9)
$$u(s,\eta) = 0 \quad , \quad (s,\eta) \in R^+ \times \Gamma \quad ,$$

$$\partial_i u_i(s,\eta) = 0 \quad , \quad (s,\eta) \in R^+ \times \Omega \quad .$$

These conditions require the "linearized fluid" to have <u>always</u> occupied the
container and to have <u>always</u> been incompressible. Using a different space
and topology, Slemrod (1976) was then able to prove that his formal equations
led to a stable linear dynamical system, paralleling our Theorem 3.1. He
also obtained an asymptotic stability result under the additional assumption
that $\int_0^r s^2 m(s)ds < \infty$, which does not presume a finite memory length r. Our
Theorem 3.2 assumes $r < \infty$, but does not require the additional assumption (3.9).

Under further assumptions on the behavior of the material function m(s),
Slemrod (1978) proved an exponential stability result. We will now present
a result on exponential decay, in our topology, for those initial
states in $\mathcal{X}$ that happen to satisfy (3.9). To this end, let $\mathcal{G}$ denote the
$\|\cdot\|_{\mathcal{X}}$ - completion of the set $\{(v,u) \in S | (3.9) \text{ holds}\}$.

<u>Theorem 3.3</u>: Let m(s) satisfy (2.3) and let there exist $\xi_2 \geq \xi_1 > 0$ such that

$$\xi_1 m(s) \leq - m'(s) \leq \xi_2 m(s) \quad \text{for all} \quad s \in [0,r) \quad .$$

Then there exist $M > 0$, $\epsilon > 0$, such that $\|T(t)x\|_{\mathcal{X}} \leq Me^{-\epsilon t}\|x\|_{\mathcal{X}}$ for all
$t \in R^+$, $x \in \mathcal{G}$.

<u>Proof:</u> It is not difficult to see that $\mathcal{G}$ is positive invariant under $\{T(t)\}_{t \geq 0}$; i.e., $\gamma(x) \subseteq \mathcal{G}$ for $x \in \mathcal{G}$. Defining $U : \mathcal{X} \to \mathcal{R}$ as

$$U(x) \equiv \|x\|_{\mathcal{X}}^2 + \beta \int_\Omega v_i(\eta) \int_0^r m(s) u_i(s, \eta) ds \, d\Omega \quad , \quad x \in \mathcal{X} \quad ,$$

where $\beta > 0$, we find that, for $x \in \mathcal{G}$,

$$\dot{U}(x) \equiv \lim_{t \searrow 0} \inf \frac{1}{t} \left[U(T(t)x) - U(x) \right]$$

$$= - \alpha \beta \int_\Omega v_i(\eta) v_i(\eta) d\Omega + \beta \int_\Omega v_i(\eta) \int_0^r m'(s) u_i(s, \eta) ds \, d\Omega$$

$$+ \beta \int_\Omega \left(\int_0^r m(s) \partial_j u_i(s, \eta) ds \right) \left(\int_0^r m(s) \partial_j u_i(s, \eta) ds \right) d\Omega$$

$$+ \int_\Omega \int_0^r m'(s) \partial_j u_i(s, \eta) \cdot \partial_j u_i(s, \eta) ds \, d\Omega \quad ,$$

where $\alpha \equiv \int_0^r m(s) ds$.

We wish to show the existence of numbers $c_1 \geq c_2 > 0$, $\epsilon > 0$, such that $c_1 \|x\|_{\mathcal{X}}^2 \geq U(x) \geq c_2 \|x\|^2$ and $\dot{U}(x) \leq - 2\epsilon U(x)$ for all $x \in \mathcal{G}$. We first notice that, by Schwarz' inequality,

$$\left| \int_0^r m(s) \partial_j u_i(s, \eta) ds \right|^2 \leq \alpha \int_0^r m(s) \partial_j u_i(s, \eta) \cdot \partial_j u_i(s, \eta) ds \quad .$$

Also, for all $z \in (\mathcal{C}_o^\infty(\Omega))^3$ it is known that

$$\int_\Omega z_i(\eta) z_i(\eta) d\Omega \leq k \int_\Omega \partial_j z_i(\eta) \cdot \partial_j z_i(\eta) d\Omega$$

for some $k(\Omega) < \infty$; hence, for $(v, u) \in \mathcal{G}$,

$$\left| \int_\Omega v_i(\eta) \int_0^r m'(s) u_i(s,\eta) \, ds \, d\Omega \right|^2 \le \left| \int_\Omega \int_0^r m'(s) v_i(\eta) v_i(\eta) \, ds \, d\Omega \right| \cdot \left| \int_\Omega \int_0^r m'(s) u_i(s,\eta) u_i(s,\eta) \, ds \, d\Omega \right|$$

$$\le m(0)k \int_\Omega v_i(\eta) v_i(\eta) \, d\Omega \cdot \left| \int_\Omega \int_0^r m'(s) \partial_j u_i(s,\eta) \cdot \partial_j u_i(s,\eta) \, ds \, d\Omega \right|$$

$$\le m(0)k\xi_2 \int_\Omega v_i(\eta) v_i(\eta) \, d\Omega \cdot \int_\Omega \int_0^r m(s) \partial_j u_i(s,\eta) \cdot \partial_j u_i(s,\eta) \, ds \, d\Omega \quad ,$$

where we have applied Schwarz' inequality. Similarly, for $x \in \mathcal{G}$,

$$\left| \int_\Omega v_i(\eta) \int_0^r m(s) u_i(s,\eta) \, ds \, d\Omega \right|^2 \le \alpha k \int_\Omega v_i(\eta) v_i(\eta) \, d\Omega \cdot \int_\Omega \int_0^r m(s) \partial_j u_i(s,\eta) \cdot \partial_j u_i(s,\eta) \, ds \, d\Omega \quad .$$

It then follows that

$$\left| U(x) - \|x\|_{\chi}^2 \right| \le (\beta \alpha k/2) \|x\|_{\chi}^2 \quad ,$$

$$\dot{U}(x) \le - \alpha \beta \int_\Omega v_i(\eta) v_i(\eta) \, d\Omega$$

$$+ \beta (k\xi_2 m(0))^{\frac{1}{2}} \left(\int_\Omega v_i(\eta) v_i(\eta) \, d\Omega \right)^{\frac{1}{2}} \left(\int_\Omega \int_0^r m(s) \partial_j u_i(s,\eta) \cdot \partial_j u_i(s,\eta) \, ds \, d\Omega \right)^{\frac{1}{2}}$$

$$- (\xi_1 - \alpha \beta) \int_\Omega \int_0^r m(s) \partial_j u_i(s,\eta) \cdot \partial_j u_i(s,\eta) \, ds \, d\Omega \quad ,$$

for all $x \in \mathcal{G}$. Choosing $\beta > 0$ so small that $\beta \alpha k < 2$ and $\beta(k\xi_2 m(0) + 4\alpha^2) < 4\alpha\xi_1$, we see that suitable numbers c_1, c_2, ϵ do exist. As $\mathcal{G}$ is positive invariant, it follows that $U(T(t)x) \le e^{-2\epsilon t} U(x)$ for all $t \in \mathbb{R}^+$, $x \in \mathcal{G}$; hence, we find that

$$\|T(t)x\|_{\chi}^2 \le (c_1/c_2) e^{-2\epsilon t} \|x\|_{\chi}^2 \quad , \quad t \in \mathbb{R}^+ \quad , \quad x \in \mathcal{G} \quad ,$$

and the proof is complete.

In this section our principal purpose was to prove that the linear abstract evolution equation (3.4) generates a dynamical system on $\mathfrak{X}$. We have gone beyond this objective, considering stability properties and exponential decay of motions, in order to provide a basis for comparison with the related results of Slemrod (1976), (1978).

<u>4. The Nonlinear Problem</u>

We now return to the nonlinear problem described by the abstract evolution equation (2.19),

$$(4.1) \qquad \dot{x}(t) = Nx(t) \quad \text{a.e.} \quad t \in \mathbb{R}^+ \quad,$$

$$x(0) = x_o \in \mathcal{X} \quad,$$

where N is the closed, densely defined operator described in §2. Questions regarding existence and uniqueness of solutions of (4.1) are quite difficult to resolve, and we are not able to prove that (4.1) is associated with a dynamical system on $\mathcal{X}$. However, assuming that it is, we shall show in this section that all positive orbits are bounded and the equilibrium at $x = 0$ is stable.

In §3 we showed that linearization of the problem led to a dynamical system on the Hilbert space $\mathcal{X}$. This suggests that it is plausible to assume that the nonlinear problem also is associated with a dynamical system on $\mathcal{X}$, in the following sense.

<u>Assumption 4.1</u>: For all sufficiently small $\lambda > 0$, $\mathbb{R}(I - \lambda N) = \mathcal{X}$ and $J_\lambda \equiv (I - \lambda N)^{-1}$ exists; moreover, with $J_0 \equiv I$, there exists a dynamical system $\{S(t)\}_{t \geq 0}$ on $\mathcal{X}$ such that $J_{t/n}^n x \to S(t)x$ as $n \to \infty$, for every $x \in \mathcal{X}$, $t \in \mathbb{R}^+$, the convergence being uniform on compact subsets of $\mathbb{R}^+$.

This particular association between the dynamical system $\{S(t)\}_{t \geq 0}$ and the operator N is motivated by considering a backward-difference approximation of equation (4.1) given by

$$x\left(\frac{mt}{n}\right) - x\left(\frac{mt}{n} - \frac{t}{n}\right) = \frac{t}{n}\,Nx\left(\frac{mt}{n}\right) \quad , \quad m = 1, 2, \cdots, n \quad ; \quad t \in \mathcal{R}^+ \quad ,$$

$$x(0) = x_o \in \mathcal{X} \quad .$$

It is seen that for sufficiently large n, depending on t, this equation has a solution $x(t) = J^n_{t/n}x_o$ if $\mathcal{R}(I - \lambda N) = \mathcal{X}$ and if $J_\lambda \equiv (I - \lambda N)^{-1}$ exists for all sufficiently small $\lambda > 0$. Under Assumption 4.1, $J^n_{t/n}x_o$ is a Hille-type approximation of $S(t)x_o$ (see Yosida (1978)).

If $\omega I - N$ were accretive for some $\omega \in \mathcal{R}$, then the theory of Crandall and Liggett (1971) would show that Assumption 4.1 holds if and only if $\mathcal{R}(I - \lambda N) = \mathcal{X}$ for all sufficiently small $\lambda > 0$ (see Yosida (1978)). Unfortunately, $\omega I - N$ is not accretive and we are unable to prove the validity of our assumption, even if $\mathcal{R}(I - \lambda N) = \mathcal{X}$.

It is remarkable that, under Assumption 4.1, it is easy to prove that all positive orbits are bounded and the equilibrium at $x = 0$ is stable.

<u>Theorem 4.2</u>: For m(s) satisfying conditions (2.3), and under Assumption 4.1, $\|S(t)x\|_{\mathcal{X}} \leq \|x\|_{\mathcal{X}}$ for all $t \in \mathcal{R}^+$, $x \in \mathcal{X}$. Furthermore, $S(t)x \to 0$ as $t \to \infty$ if the positive orbit $\Upsilon(x)$ is precompact.

<u>Proof</u>. Defining $V(x) \equiv \|x\|_{\mathcal{X}}^2$ and

$$\dot{V}(x) \equiv \lim_{t \searrow 0} \inf \frac{1}{t} \left[V(S(t)x) - V(x)\right]$$

for $x \in \mathcal{X}$, we note that for $\lambda > 0$ and $x \in \mathcal{D}(N)$,

$$V(x - \lambda Nx) \geq V(x) - 2\lambda \langle x, Nx \rangle \quad , \quad$$

$$\langle x, Nx \rangle = - \int_\Omega v_i(\eta) \cdot \pi_{i\ell} \left\{ v_j(\eta) \cdot \partial_j v_\ell(\eta) + \partial_j \int_0^r m(s) [\partial_\ell u_k(s,\eta) \cdot \partial_j u_k(s,\eta) + \partial_j u_\ell(s,\eta)] ds \right\} d\Omega$$

$$- \int_\Omega \int_0^r m(s) \partial_k u_i(s,\eta) \cdot \partial_k \left\{ v_i(\eta) + v_j(\eta) \cdot \partial_j u_i(s,\eta) + \frac{\partial}{\partial s} u_i(s,\eta) \right\} ds \, d\Omega$$

$$= - \frac{1}{2} \int_\Omega v_j(\Omega) \cdot \partial_j \left\{ v_\ell(\eta) v_\ell(\eta) + \int_0^r m(s) \partial_k u_i(s,\eta) \cdot \partial_k u_i(s,\eta) ds \right\} d\Omega$$

$$- \int_\Omega v_\ell(\eta) \cdot \partial_j \int_0^r m(s) [\partial_\ell u_k(s,\eta) \cdot \partial_j u_k(s,\eta) + \partial_j u_\ell(s,\eta)] ds \, d\Omega$$

$$- \int_\Omega \partial_k v_j(\eta) \cdot \int_0^r m(s) [\partial_j u_i(s,\eta) \cdot \partial_k u_i(s,\eta) + \partial_k u_j(s,\eta)] ds \, d\Omega$$

$$- \frac{1}{2} \int_\Omega \int_0^r m(s) \frac{\partial}{\partial s} [\partial_k u_i(s,\eta) \cdot \partial_k u_i(s,\eta)] ds \, d\Omega$$

$$= \frac{1}{2} \int_\Omega \int_0^r m'(s) \partial_k u_i(s,\eta) \cdot \partial_k u_i(s,\eta) ds \, d\Omega \equiv - \frac{1}{2} W(x) \leq 0 \quad .$$

As $\mathfrak{D}(N)$ is dense and Assumption 4.1 is made, Theorem 3.4 of Walker (1979) shows that $\dot{V}(x) \leq - W(x) \leq 0$ for all $x \in \mathfrak{X}$; consequently, $V(S(t)x) \leq V(x)$ for all $t \in \mathfrak{R}^+$, $x \in \mathfrak{X}$. As in the proof of Theorem 3.2, we also see that the largest positive invariant set $\mathfrak{m}^+$ in $\{x \in \mathfrak{X} | \dot{V}(x) = 0\}$ is $\mathfrak{m}^+ = \{0\}$; hence, if x is such that $\gamma(x)$ is precompact, LaSalle's Invariance Principle (see Hale (1969)) implies that $S(t)x \to 0$ as $t \to \infty$. The proof is complete.

Theorem 4.2 shows that all positive orbits are bounded and the equilibrium at $x = 0$ is stable. If all positive orbits could be shown to be precompact, Theorem 4.2 would imply that $x = 0$ is globally asymptotically stable. Although we do not know that all positive orbits are precompact, the last result

of Theorem 4.2 does show that there exist no equilibria (steady flows) other than x = 0 (the rest condition), and it also shows that there are no nontrivial periodic motions (nonsteady periodic flows).

We remark that the function $V(x) \equiv \|x\|_{\mathcal{X}}^2$ used in the proof of the foregoing theorem is a Liapunov function for $\{S(t)\}_{t \geq 0}$ (see Hale (1969) and Walker (1979)). Useful Liapunov functions often are extremely difficult to discover for a highly nonlinear problem, and discovery of a topology suitable for a state space may be even more difficult. In fact, these difficulties are so inter-related (see Walker (1976)) that a "formal" Liapunov function (for the formal equation) often is sought a priori, as a means of suggesting a suitable topology for the state space. This is what led us to set equation (4.1) and Assumption 4.1 in the particular Hilbert space $\mathcal{X}$, rather than in any other metric space.

The function V is the only Liapunov function that we have been able to find for the nonlinear problem. This is in contrast with the linearized problem of §2, which admits an infinite family of useful Liapunov functions, and leads to a corresponding family of linear dynamical systems on state spaces differing in their topologies. The linear dynamical system and Liapunov function of Slemrod (1976) belong to this family, which can be defined in terms of the set of linear operators that commute with the linear operator A of (3.3). However, among all of the functions in this family, it appears that only the function V employed here is useful with the original nonlinear problem.

Our Liapunov function V admits a simple physical interpretation. To demonstrate this point most clearly, we return to the formal problem of §2 and relax certain of our assumptions. Rather than assuming that $v_i(\eta, t) = 0$ for

$(\eta, t) \in \Gamma \times \mathbb{R}^+$, let us assume only that fluid can not cross the boundary Γ of Ω; that is, $v_i(\eta, t) n_i(\eta) = 0$ for $(\eta, t) \in \Gamma \times \mathbb{R}^+$, where $n(\eta)$ denotes the unit outward normal to Γ. Let us also consider a general body force field $f_i(\eta, t)$, rather than one derived from a potential $q(\eta, t)$. If we retain all other assumptions of §2, formal computations lead to

$$\frac{d}{dt} V(x(t)) = \int_\Omega \int_0^r m'(s) \partial_k u_i(s, \eta, t) \partial_k u_i(s, \eta, t) ds \, d\Omega + 2P(t)$$

$$\leq 2P(t) \quad , \quad t \in \mathbb{R}^+ \quad ,$$

where $P(t)$ is the external power,

$$P(t) \equiv \int_\Omega f_i(\eta, t) v_i(\eta, t) d\Omega + \int_\Gamma n_i(\eta) s_{ij}(\eta, t) v_i(\eta, t) d\Gamma \quad .$$

Consequently, we see that $\frac{1}{2} V$ has the basic property of a "mechanical energy function" for the fluid. When all assumptions of §2 are applied, $P(t) = 0$ and $\frac{d}{dt} V(x(t)) \leq 0$ for all $t \in \mathbb{R}^+$.

The assumption of "fading memory" played a crucial role in our analysis; however, we remark that conditions (2.3) can be slightly weakened. It is not difficult to see that we do not actually need $m \in C^1(\mathbb{R}^+)$ and, except for Theorem 3.3, all of our theorems continue to hold if conditions (2.3) are replaced by

a) $m : \mathbb{R}^+ \to \mathbb{R}$ is integrable, with $0 < \int_0^\infty m(s) ds < \infty$,

(4.2) b) m is nonnegative and nonincreasing on $\mathbb{R}^+$,

c) $m(s) > 0$ for $s \in [0, r)$; $m(s) = 0$ for $s \notin (0, r)$,

where $r > 0$ may be infinite.

References

Coleman, B. D., and V. J. Mizel, Norms and semigroups in the theory of fading memory. Arch. Rational Mech. Anal. 23 (1966), 87-123.

Coleman, B. D., and W. Noll, Foundations of linear viscoelasticity. Rev. Mod. Phys. 33 (1961), 239-249.

Coleman, B. D., and W. Noll, Simple fluids with fading memory. Proceedings of the international symposium on second-order effects in elasticity, plasticity, and fluid dynamics; Haifa, Israel. Oxford, Paris, New York: Pergamon Press, 1964.

Craik, A. D., A note on the static stability of an elastoviscous fluid. J. Fluid Mech. 33 (1968), 33-38.

Crandall, M. G., and T. M. Liggett, Generation of semigroups of nonlinear transformations on general Banach spaces. Amer. J. Math. 93 (1971), 265-298.

Dafermos, C. M., and M. Slemrod, Asymptotic behavior of nonlinear contraction semigroups. J. Funct. Anal. 13 (1973), 97-106.

Fujita, H., and T. Kato, On the Navier-Stokes initial value problem. Arch. Rational Mech. Anal. 16 (1964), 269-315.

Hale, J. K., Dynamical systems and stability. J. Math. Anal. Appl. 26 (1969), 39-59.

Joseph, D. D., Slow motion and viscometric motion; stability and bifurcation of the rest state of a simple fluid. Arch. Rational Mech. Anal. 56 (1974), 99-157.

Mizohata, S., The theory of partial differential equations. London, New York: Cambridge U. Press (1973).

Noll, W., A mathematical theory of the mechanical behavior of continuous media. Arch. Rational Mech. Anal. 2 (1958), 197-226.

Slemrod, M., A hereditary partial differential equation with applications in the theory of simple fluids. Arch. Rational Mech. Anal. 62 (1976), 303-321.

Slemrod, M., An energy stability method for simple fluids. Arch. Rational Mech. Anal. (1978), forthcoming.

Walker, J. A., On the application of Liapunov's Direct Method to linear dynamical systems. J. Math. Anal. Appl. 53 (1976), 187-220.

Walker, J. A., Some results on Liapunov functions and generated dynamical systems. J. Diff. Eq. (1979), forthcoming.

Yosida, K., Functional analysis, New York: Springer-Verlag (1978).